ME136/236: Cheat sheet

AERODYNAMICS

Reynold's number, $Re := \frac{v_\infty l \rho}{\mu}$, Mach number, $M := \frac{v_\infty}{v_{\text{sound}}}$, with free-stream air speed v_∞.

Aerofoils:

- $f_l = \frac{1}{2}\rho v_\infty^2 c C_l$
- α: aerofoil angle of attack [rad]
- Thin aerofoil model:
  - Lift curve slope: $\frac{\partial C_l}{\partial \alpha} = 2\pi$
  - $C_l = C_{l_0} + \frac{\partial C_l}{\partial \alpha}\alpha = C_{l_0} + 2\pi\alpha$

Finite wings:

- Lift: $f_L = \frac{1}{2}\rho v_\infty^2 S C_L$:
  - S: Surface area
  - c: chord (wing width, varies along wing)
  - b: span (wing tip-to-tip distance)
  - AR: aspect ratio, $AR = \frac{b^2}{S}$
  - Analytical model for a finite wing, flat plate aerofoil: $C_L = \frac{AR}{AR+2} C_l$
  - Induced drag: $f_{D_i} := f_L \sin\alpha_i = \frac{1}{2}\rho v_\infty^2 S C_{D_i}$
    Analytical model: $C_{D_i} \approx \frac{C_L^2}{\pi AR}$

Drag: Form drag (pressure drag):

$$f_D = \frac{1}{2}\rho v_\infty^2 S C_D$$

Propellers Thrust f_T [N] from scalar speed Ω [rads^{-1}]

$f_T = C_T \Omega^2$, Torque: $\tau = \gamma f_T$

Momentum theory: propeller radius r_p, power consumed P:

$$P = \frac{f_T^{3/2}}{r_p\sqrt{2\pi\rho}}$$

TENSORS & DYNAMICS

- Displacement of point A w.r.t. point B: $\underline{s_{AB}}$
- Velocity of point A w.r.t. frame E (B is fixed in E), $\underline{v_A^E} := D^E \underline{s_{AB}}$: $\underline{v_A^E}$
- Rotation of frame B w.r.t. frame A: $\underline{R^{BA}}$
- Ang. velocity of frame B w.r.t. frame A: $\underline{\omega^{BA}}$
- (capital letter) Skew symmetric form of $\underline{\omega^{BA}}$ is $\underline{\Omega^{BA}}$:
- Identity tensor (identity matrix in any coord. sys:) $\underline{I}$
- Overbar is 'transpose': $\overline{\underline{x}}$

Skew-symmetric form:

$$\text{if } [\underline{x}]^A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ then } [\underline{X}]^A = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Changing coordinate systems: $[T]^{AB}$ transforms to A w.r.t. B.

Definition of tensors: for any coordinate systems A, B:

- First order tensor $\underline{x}$: $[\underline{x}]^A = [T]^{AB}[\underline{x}]^B$
- Second order tensor $\underline{Y}$: $[\underline{Y}]^A = [T]^{AB}[\underline{Y}]^B\overline{[T]}^{AB}$

Contraction of indices:

- Displacements: $\underline{s_{AC}} = \underline{s_{AB}} + \underline{s_{BC}}$
- Transformations: $[T]^{AC} = [T]^{AB}[T]^{BC}$

Length of a tensor:

- $\|\underline{x}\| = \sqrt{\overline{\underline{x}}\underline{x}} = \sqrt{\overline{[\underline{x}]}^A[\underline{x}]^A} = \sqrt{x_1^2+x_2^2+x_3^2}$ for any coordinate system A, and wherein $(x_1,x_2,x_3) = [\underline{x}]^A$.

Transformation matrix:

$$[T]^{BA} = \begin{bmatrix} [\underline{1^A}]^B & [\underline{2^A}]^B & [\underline{3^A}]^B \end{bmatrix} = \begin{bmatrix} \overline{[\underline{1^B}]}^A \\ \overline{[\underline{2^B}]}^A \\ \overline{[\underline{3^B}]}^A \end{bmatrix}$$

Euler angles: Encode transformation matrix $[T]^{BE}$ with three parameters: $[T]^{BE} = [T]^{BY}[T]^{YX}[T]^{XE}$:

- Yaw (ψ): rotate E about $\underline{3^E} = \underline{3^X}$ to generate X system
- Pitch (θ): rotate X about $\underline{2^X} = \underline{2^Y}$ to generate Y system
- Roll (ϕ): rotate Y about $\underline{1^Y} = \underline{1^B}$ to generate B system

$$[T]^{BE} = \begin{bmatrix} \mathrm{c}\psi\mathrm{c}\theta & \mathrm{s}\psi\mathrm{c}\theta & -\mathrm{s}\theta \\ \mathrm{c}\psi\mathrm{s}\theta\mathrm{s}\phi - \mathrm{s}\psi\mathrm{c}\phi & \mathrm{s}\psi\mathrm{s}\theta\mathrm{s}\phi + \mathrm{c}\psi\mathrm{c}\phi & \mathrm{c}\theta\mathrm{s}\phi \\ \mathrm{c}\psi\mathrm{s}\theta\mathrm{c}\phi + \mathrm{s}\psi\mathrm{s}\phi & \mathrm{s}\psi\mathrm{s}\theta\mathrm{c}\phi - \mathrm{c}\psi\mathrm{s}\phi & \mathrm{c}\theta\mathrm{c}\phi \end{bmatrix}$$

Rotation tensor For frames A, B: $\underline{1^A} = \underline{R^{AB}}\,\underline{1^B}$ (same for $\underline{2^A}$ and $\underline{3^A}$).

$$[\underline{R^{BA}}]^B = [\underline{R^{BA}}]^A = \overline{[T]}^{BA} = [T]^{AB}$$

Rotational time derivative: (Note: $\left[\frac{\mathrm{d}}{\mathrm{d}t}\underline{x}\right]^A \neq [T]^{AB}\left[\frac{\mathrm{d}}{\mathrm{d}t}\underline{x}\right]^B$ in general.)

$$[\underline{D^A x}]^B := \left[\frac{\mathrm{d}}{\mathrm{d}t}\underline{x}\right]^B + [T]^{BA}\overline{\left[\frac{\mathrm{d}}{\mathrm{d}t}T\right]}^{BA}[\underline{x}]^B$$

Preserves $[\underline{D^C x}]^A = [T]^{AB}[\underline{D^C x}]^B$ for any A, B, C.

When superscripts match:

$$[\underline{D^A x}]^A = \left[\frac{\mathrm{d}}{\mathrm{d}t}\underline{x}\right]^A = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} \text{ for } [\underline{x}]^A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Euler transformation: $D^A\underline{x} = D^B\underline{x} + \underline{\Omega^{BA}}\,\underline{x}$

- Translational velocity: $\underline{v_B^A} := D^A \underline{s_{BA_1}}$ where A_1 fixed w.r.t. frame A.
- Translational acceleration: $\underline{a_B^A} := D^A \underline{v_B^A}$.
- Angular velocity: $\underline{\Omega^{BA}} := \left(D^A \underline{R^{BA}}\right)\overline{\underline{R^{BA}}}$ (second order tensor).
  - $\underline{\Omega^{BA}} = -\underline{\Omega^{AB}} = \overline{\underline{\Omega^{AB}}}$
  - Related first order tensor is $\underline{\omega^{BE}}$ (if B is body-fixed, E is earth-fixed):

$$[\underline{\omega^{BE}}]^B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}; \quad [\underline{\Omega^{BE}}]^B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}T\right]^{EB} = [T]^{EB}[\underline{\Omega^{BE}}]^B$$

Time derivative of Euler angles:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

For small angles only:

$\frac{\mathrm{d}}{\mathrm{d}t}(\phi,\theta,\psi) \approx (p,q,r)$

Center of mass: That point C for which

$$\sum_i m^i \underline{s_{P_iC}} = 0$$

Last update: 2019-09-11 at 14:11:24

Translational momentum:

$$\underline{p}_C^E := m^B \underline{v}_C^E$$

Newton's law:

$$D^I \underline{p}_C^I = \underline{f}$$

Inertia tensor: Body B, reference point R: ($\underline{S_{P_iR}}$ is skew-symmetric form of $\underline{s_{P_iR}}$)

$$\underline{J}_R^B := \sum_i m^i \overline{\underline{S_{P_iR}}}\,\underline{S_{P_iR}}$$

(See expansion at end of cheat-sheet)

Angular momentum: For ref. point equal to center of mass:

$$\underline{l}_B^{BA} = \underline{J}_B^B \underline{\omega}^{BA}$$

Moment of force ("torque"):

$$\underline{n}_R = \underline{S_{PR}}\,\underline{f}$$

Euler's law:

$$D^I \underline{l}_B^{BI} = \underline{n}_B$$

For a rigid body, coordinated in B:

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\underline{\omega}^{BE}\right]^B = \left(\left[\underline{J}_B^B\right]^B\right)^{-1}\left(\left[\underline{n}_B\right]^B - \left[\underline{\Omega}^{BE}\right]^B\left[\underline{J}_B^B\right]^B\left[\underline{\omega}^{BE}\right]^B\right)$$

SENSORS

Accelerometer: Output is $[\underline{\alpha}]^S$

$$[\underline{\alpha}]^S = \left[\underline{a}_B^E\right]^S - [\underline{g}]^S$$

Rate gyroscope: Output is $[\underline{\gamma}]^S$

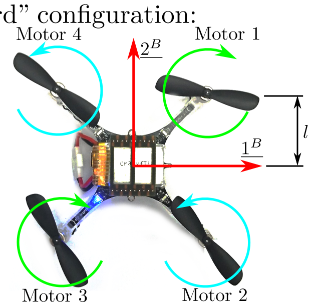
$$[\underline{\gamma}]^S = \left[\underline{\omega}^{BE}\right]^S$$

Optical flow: Output is (σ_1,σ_2)

$$\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\left(-\frac{1}{\|\underline{s_{FS}}\|}\left[\underline{v}_S^E\right]^S + \left[\underline{\Omega}^{BE}\right]^S\left[\underline{3}^S\right]^S\right)$$

QUADCOPTER DYNAMICS

"ME136 Standard" configuration:



$$\begin{bmatrix}c_\Sigma\\ n_1\\ n_2\\ n_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\ l & -l & -l & l\\ -l & -l & l & l\\ \kappa & -\kappa & \kappa & -\kappa\end{bmatrix}\begin{bmatrix}c_{P_1}\\ c_{P_2}\\ c_{P_3}\\ c_{P_4}\end{bmatrix}$$

$$\begin{bmatrix}c_{P_1}\\ c_{P_2}\\ c_{P_3}\\ c_{P_4}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1 & l^{-1} & -l^{-1} & \kappa^{-1}\\ 1 & -l^{-1} & -l^{-1} & -\kappa^{-1}\\ 1 & -l^{-1} & l^{-1} & \kappa^{-1}\\ 1 & l^{-1} & l^{-1} & -\kappa^{-1}\end{bmatrix}\begin{bmatrix}c_\Sigma\\ n_1\\ n_2\\ n_3\end{bmatrix}$$

DESCRIPTIONS OF ROTATIONS

Rotation vector, axis-angle: Rotate from frame A about axis $\underline{n}^{BA}$ by an angle ρ^{BA} to generate frame B. Alternatively, rotation vector $\underline{\lambda}^{BA} := \rho^{BA}\underline{n}^{BA}$.

$$[T]^{BA} = \exp\left(-\left[\underline{\Lambda}^{BA}\right]^B\right)$$

$$= [\underline{I}] + \left(-\left[\underline{\Lambda}^{BA}\right]^B\right) + \frac{1}{2!}\left(-\left[\underline{\Lambda}^{BA}\right]^B\right)^2 + \frac{1}{3!}\left(-\left[\underline{\Lambda}^{BA}\right]^B\right)^3 \ldots$$

$$= \cos\rho^{BA}[\underline{I}] + \left(1-\cos\rho^{BA}\right)\left[\underline{n}^{BA}\right]^B\overline{\left[\underline{n}^{BA}\right]}^B - \sin\rho^{BA}\left[\underline{N}^{BA}\right]^B$$

Euler Symmetric Parameters:

$$\{q\}^{BA} := \begin{bmatrix}\cos\left(\frac{1}{2}\rho^{BA}\right)\\ \left[\underline{n}^{BA}\right]^B\sin\left(\frac{1}{2}\rho^{BA}\right)\end{bmatrix} =: \begin{bmatrix}q_0\\ [\underline{q_v}]\end{bmatrix} = (q_0,q_1,q_2,q_3)$$

$$[T]^{BA} = q_0^2[\underline{I}] - \overline{[\underline{q_v}]}\,[\underline{q_v}]\,[\underline{I}] + 2[\underline{q_v}]\,\overline{[\underline{q_v}]} - 2q_0[\underline{Q_v}]$$

$$= \begin{bmatrix}q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_0q_3) & 2(q_1q_3-q_0q_2)\\ 2(q_1q_2-q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_0q_1)\\ 2(q_1q_3+q_0q_2) & 2(q_2q_3-q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2\end{bmatrix}$$

$$\{q\}^{CA} = \overline{\{Q\}}^{CB}\{q\}^{BA}$$

$$\{Q\}^{BA} = q_0\{I\} + \begin{bmatrix}0 & \overline{[\underline{q_v}]}\\ -[\underline{q_v}] & [\underline{Q_v}]\end{bmatrix}$$

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t}q\right\}^{BA} = \frac{1}{2}\overline{\{\Omega\}}^{BA}\{q\}^{BA};\ \text{with } \{\omega\}^{BA} = \begin{bmatrix}0\\ \left[\underline{\omega}^{BA}\right]^B\end{bmatrix}$$

CONTROL

Canonical model: $\dot{x} = f(x,u)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$.

Equilibrium: x^*, u^* such that $\dot{x} = 0$

First order system: $\dot{x} = -\frac{1}{\tau}x$

Second order, damped system: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$

$$\left[\underline{J}_R^B\right]^B := \begin{bmatrix}\sum_i m^i\left(\left(s_{P_iR}^2\right)^2+\left(s_{P_iR}^3\right)^2\right) & -\sum_i m^i s_{P_iR}^1 s_{P_iR}^2 & -\sum_i m^i s_{P_iR}^1 s_{P_iR}^3\\ -\sum_i m^i s_{P_iR}^1 s_{P_iR}^2 & \sum_i m^i\left(\left(s_{P_iR}^1\right)^2+\left(s_{P_iR}^3\right)^2\right) & -\sum_i m^i s_{P_iR}^2 s_{P_iR}^3\\ -\sum_i m^i s_{P_iR}^1 s_{P_iR}^3 & -\sum_i m^i s_{P_iR}^2 s_{P_iR}^3 & \sum_i m^i\left(\left(s_{P_iR}^1\right)^2+\left(s_{P_iR}^2\right)^2\right)\end{bmatrix}$$